Quantum diffusion in a biased kicked Harper system

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Quantum diffusion in a biased kicked Harper system, modeling field-induced transport in superlattices, is studied for fully chaotic dynamics of the underlying classical system. Under these conditions, the classical transport is diffusive whereas the quantum diffusion can be either enhanced or suppressed for commensurable or incommensurable ratio of the Bloch period to the driving period, respectively. The quantum transport properties are related to the statistical properties of the quasienergy spectra as described by random matrix theory.

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I. INTRODUCTION

Recently much attention has been paid to the transport and spectral properties of quantum periodic structures with classically chaotic dynamics [1–6]. For example, Ref. [1] discusses the dynamics of a quantum particle in an amplitude-modulated cosine potential, Ref. [2] deals with a periodic chain of chaotic billiards, and Ref. [3] focuses on the effect of classical stability islands, etc. In the present paper we study the process of "quantum diffusion" in periodically driven biased superlattices. We show that this problem has many similarities with the famous quantum kicked rotor problem [7]. Namely, the temporal regime of classical diffusion is changed to quantum ballistic or saturation regimes, depending on the value (rational or irrational) of some control parameter. The relation of these two regimes of quantum dynamics to the quasienergy spectrum of the system is discussed and it is shown that the superdiffusion or the suppressed diffusion can be explained by the statistical properties of the spectrum, more precisely its analysis by random matrix theory (RMT).

II. THE BIASED HARPER MODEL

The notion of a superlattice implies that in addition to the natural (lattice) period, there is a larger period in the system, enforced by some periodic potential $V(x)$. (One may think, e.g., of a semiconductor superlattice as a physical object.) Because of the presence of this potential, each Bloch band of the system is split into N minibands, where N is the superlattice period in units of the lattice period. In what follows, we shall use scaled variables, where the superlattice period is taken as 2π. Then, restricting the analysis to the ground (original) Bloch band and using the basis of localized Wannier states (which are essentially localized within one well of the lattice potential and provide an alternative basis in the Hilbert space spanned by the Bloch functions) the Hamiltonian of the biased superlattice, i.e., a superlattice in an external field, takes the form

$$\hat{H}=-\cos\hat{p}+V(x)+Fx,\quad V(x+2\pi)=V(x), \tag{1}$$

where $\hat{p}=-i\hbar\partial/\partial x$ and $\hbar=2\pi/N$. It is worth noting that by construction the coordinate x in the Hamiltonian (1) is a discrete variable, i.e., $x\equiv x_n=2\pi n/N$. Correspondingly, the operator $\cos\hat{p}$ is the sum of shift operators over the lattice period $2\pi/N$, i.e., $\cos\hat{p}\,\psi(x_n)=[\psi(x_{n+1})+\psi(x_{n-1})]/2$. (Note that this leads to the celebrated tight-binding model.)

The classical counterpart of the Hamiltonian (1) obviously reads as $H=-\cos p+V(x)+Fx$, where the (continuous) variables x $(-\infty<x<\infty)$ and p $(-\pi<p\leqslant\pi)$ are the coordinate and momentum of the classical particle. It is easy to see that the motion of a classical particle with this Hamiltonian is always bounded to a region $\sim 1/F$. However, if we periodically drive the system, $V(x)\to V(x,t)=V(x,t+T)$, the particle can go arbitrarily far from its initial location. A prerequisite for this is the chaotic dynamics of the system in the absence of the static field. Because the particular form of $V(x,t)$ is irrelevant to the effects discussed below, we shall use for simplicity a δ-kicked harmonic perturbation. Thus our model corresponds to a biased kicked Harper system,

$$\hat{H}=-\cos\hat{p}-K\cos x\sum_m \delta(t-mT)+Fx. \tag{2}$$

It is worth noting that, since the kicked Harper system was introduced in Ref. [8] to model a random walk on a web and in Refs. [9,10] in the field of classical and quantum chaos, it has been widely used to model different phenomena and may appear in very different context (see, for example, Refs. [4,5], which also contain an extended list of references to kicked Harper related problems). A stroboscopic bias has been considered in the quantum kicked rotor to follow accelerator mode islands [11]. In the present work we use it to study the effect of a static field on the chaotic diffusion of the *coordinate*. To simplify the analysis, we assume that for $F=0$ the kicked Harper system is in the regime of fully de-

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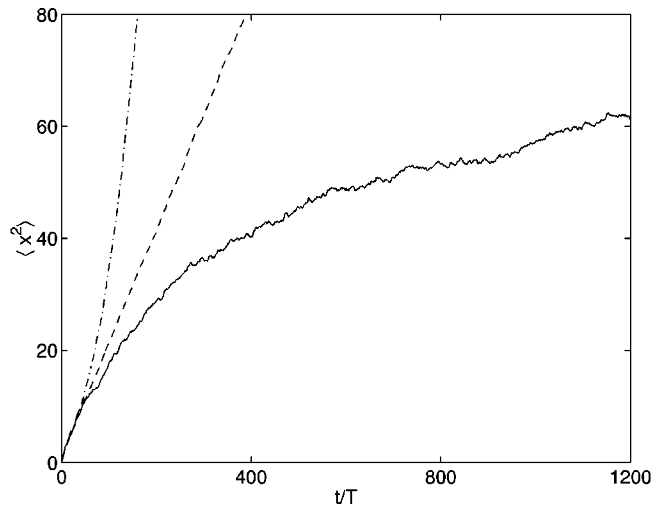

FIG. 1. Diffusion regimes for the biased Harper system (2). The dashed line shows the expectation value $\langle x^2\rangle$ for the classical diffusion (system parameters $K=2$, $T=4$, and $F=0.024$). The corresponding quantum evolution ($\hbar=2\pi/63$) shows a suppression of the classical diffusion (solid line), whereas in the commensurate case, $T_B=T$ (which is realized by choosing $F=\hbar/T\approx0.024\,93$) the quantum diffusion is accelerated (dashed-dotted line).

veloped chaos (no stability islands), and we restrict ourselves to such a weak field F that it does not change this property of the system. (In general, the effect of the static field is rather nontrivial and it may both enhance or suppress chaos.) Under this presumption, the classical system (2) shows perfect diffusion, i.e., the mean squared displacement grows asymptotically linearly in time, $\langle x^2\rangle=Dt$, with essentially the same value of the diffusion constant D as in the unbiased case. This is illustrated by the dashed line in Fig. 1 for system parameters $K=2$, $T=4$, $F=0.024$ and an initial "minimum uncertainty" ensemble of classical particles with $\langle x\rangle=\langle p\rangle=0$. (The diffusion constant $D\approx0.05$ as to be compared to $D\approx0.07$ for $F=0$.)

The quantum dynamics in the presence of static field is, however, quite different from the classical one. The key point is that the biased quantum system has another (intrinsic) time period besides the period of the driving force T. This is the so-called Bloch period $T_B=\hbar/F$. Figure 1 also shows $\langle x^2\rangle$ for the quantum evolution. Depending on the ratio T/T_B, the temporal classical diffusion becomes either accelerated (dashed-dotted line) or suppressed (full line). Namely, for a commensurable ratio, one finds a ballistic dynamics with $\langle x^2\rangle\sim t^2$ for large times, and for an irrational ratio the classical diffusion is quantum mechanically localized. These differences in the diffusion regimes reflect the qualitatively different structures of the quasienergy spectra in the two cases, which are analyzed in the following.

III. THE QUASIENERGY SPECTRUM

We begin with the analysis of the quasienergy spectrum of the biased kicked Harper system (2) for resonant driving, $T=T_B$. In this case, the quasienergy spectrum $\epsilon_\alpha(\kappa)$ of the system determined by the eigenstates of the Floquet operator,

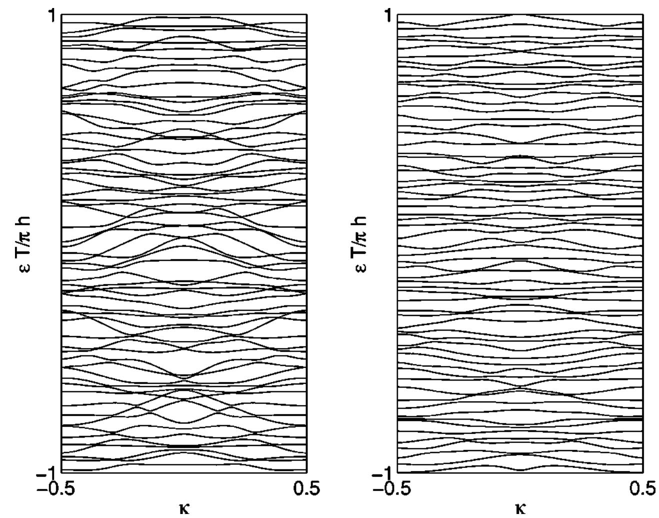

FIG. 2. The quasienergy spectrum of the system for $F=0$ (left panel) and $F=\hbar/T$ (right panel) ($K=2$, $T=4$, $N=2\pi/\hbar=63$).

$$\hat U(T)\psi_{\alpha,\kappa}(x)=\exp\left(-\frac{i}{\hbar}\epsilon_\alpha(\kappa)T\right)\psi_{\alpha,\kappa}(x), \tag{3}$$

consists of N bands $\epsilon_\alpha(\kappa)$, $\alpha=1,\dots,N$, with a quasimomentum κ confined to the first Brillouin zone $-1/2<\kappa\leqslant1/2$. To find the spectrum numerically, one has to diagonalize a κ-dependent $N\times N$ unitary matrix given by the expression [12]

$$\hat U^{(\kappa)}(T)=\exp\left(\frac{iFT}{\hbar}x\right)\widehat{\exp}\left(\frac{i}{\hbar}\int_0^T\cos(\hat p+\hbar\kappa-Ft)dt\right)\times\exp\left(\frac{iK}{\hbar}\cos x\right), \tag{4}$$

where $\widehat{\exp}$ denotes time ordering. We note that due to the condition $T=T_B$, the quantity $FT/\hbar$ in the first exponent equals unity. The advantage of the displayed form of the evolution operator is that it also correctly captures the case of a vanishing static field.

The spectra of the system in these two cases ($F=0$ and $F=\hbar/T$) are compared in Fig. 2 for $K=2$, $T=4$, and $N=2\pi/\hbar=63$. Even visually it can be concluded that the static field changes the statistical properties of the spectrum. Indeed, it can be shown that for $F\neq0$ and any value of κ, excluding $\kappa=0$ and $\kappa=\pm1/2$, matrix (4) has no antiunitary symmetry [13,14]. Thus, for $F\neq0$, the statistical properties of the spectrum should be similar to those of a circular unitary ensemble (CUE) of random matrices, whereas for $F=0$ we know that the statistics follows the circular orthogonal ensemble (COE) [15]. In particular, we expect that the distribution of the gaps $s=(\epsilon_{\alpha+1}-\epsilon_\alpha)/\Delta$, where Δ is the mean gap width, follows the RMT distributions

$$P_{\mathrm{COE}}(s)=\frac{\pi}{2}s\,e^{-\pi s^2/4}, \tag{5}$$

for $F=0$, and

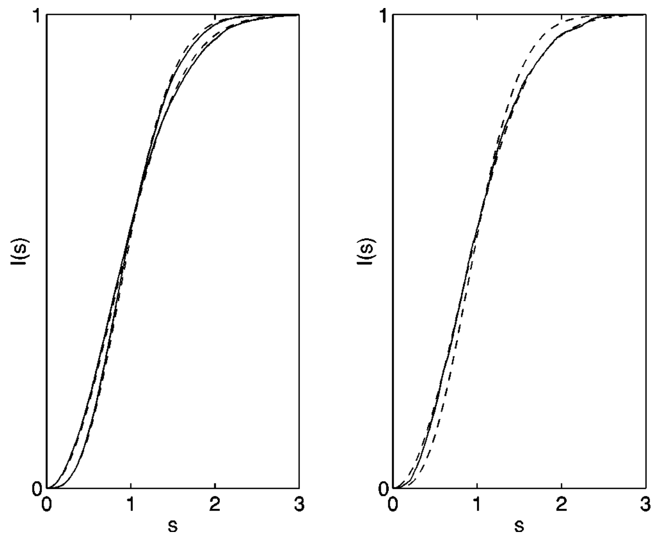

FIG. 3. Integrated distribution of the energy gaps. The two solid lines in the left panel are the numerical results for $F=0$ and $F=\hbar/T$, the solid line in the right panel corresponds to $F=29\hbar/27T$. (Scaled Planck's constant $\hbar=2\pi/255$, the other parameters are the same as in Fig. 2). The dashed lines show COE and CUE statistics according to RMT.

$$P_{\mathrm{CUE}}(s)=\frac{32}{\pi^2}s^2e^{-4s^2/\pi}, \tag{6}$$

for $F\neq0$. [In Eqs. (5) and (6) the condition $N\gg1$ is assumed.] We shall test our conjecture in terms of the cumulated distributions $I(s)=\int_0^s P(s')\,\mathrm{d}s'$. Then Eq. (5) and Eq. (6) take the form $I_{\mathrm{COE}}(s)=1-\exp(-\pi s^2/4)$ and $I_{\mathrm{CUE}}(s)=\mathrm{erf}(2s/\sqrt{\pi})-(4s/\pi)\exp(-4s^2/\pi)$, respectively. The comparison of the cumulated distributions of the energy gaps, shown in the left panel of Fig. 3, confirms the anticipated statistics. (In order to increase the number of states, $N=255$ has been used instead of $N=63$ in Fig. 2.) It is clearly seen that for the biased system with F chosen to satisfy the condition $T_B=T$, the statistics of the quasienergy gaps changes from COE to CUE. Let us also note that the coincidence between numerical data and theoretical curves can be improved if the vicinities $\delta\kappa\sim\hbar$ of the center and the edges of Brillouin zone are excluded from the analysis. These are the so-called "contact zones," where the symmetry, and hence the adequate RMT ensemble, changes (see, e.g., the related observations in studies of magnetic band structures [16] or billiard chains [6]).

Next we analyze the case $T_B/T=r/q$ where r and q are coprime integers. It has been shown in Ref. [12] that in this case the quasienergy spectrum of the system can be constructed from the spectrum of the evolution operator over the *common* period $rT=qT_B$,

$$\hat{U}(rT)=\exp(iqx)\prod_{n=1}^{r}\left[\widehat{\exp}\left(\frac{i}{\hbar}\int_{(n-1)T}^{nT}\cos(p-Ft)dt\right)\times\exp\left(\frac{iK}{\hbar}\cos x\right)\right]. \tag{7}$$

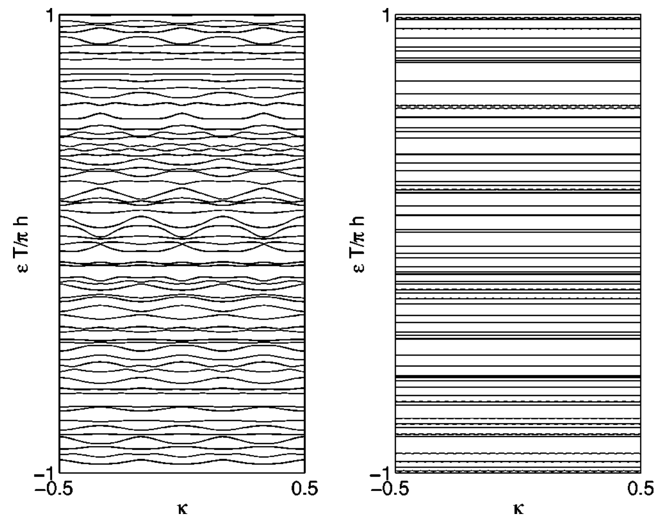

FIG. 4. The spectrum of the evolution operator (7) for $T_B/T=r/q$ with $r=3$, $q=4$ (left panel), and $r=27$, $q=29$ (right panel).

The energy bands of operator (7) are periodic functions of the quasimomentum with period $1/r$ (see Fig. 4). This reflects the fact that the quasienergy spectrum of system (2) is confined to the reduced Brillouin zone $-1/2r<\kappa\leq1/2r$. It is also interesting to note that for large values of r and q the distribution of the energy gaps again approaches a COE statistics. This can be understood by noticing that for a small Brillouin zone (which is the case for a large r) any point is actually in the "vicinity" of the special points $\kappa=0$ (zone center) or $\kappa=\pm1/2r$ (zone edges). As an example, the right panel of Fig. 3 shows the integrated gap statistics for a slightly increased field $F=29\hbar/27T\approx1.07\hbar/T$ compared to $F=\hbar/T$ in the left panel. We clearly see that the COE distribution of the unbiased case is approximately restored.

The other important result, which can be concluded from Fig. 4, is that the widths of the quasienergy bands tend to zero when $r,q\to\infty$. This is in agreement with the assumption about the discrete quasienergy spectrum in the case of incommensurate periods. (Although this assumption looks quite natural, we are not aware of any formal proof of this statement.) In principle, one can study the dependence of the bandwidth on the system parameters. However, the more important quantity for our present aim is the mean squared group velocity

$$v=\frac{T}{r}\left[\left\langle\left(\frac{\partial\epsilon_\alpha(\kappa)}{\hbar\,\partial\kappa}\right)^2\right\rangle\right]^{1/2}, \tag{8}$$

where the angle brackets denote an average over the band index and the quasimomentum. It is easy to show that quantity (8) is directly related to the prefactor of the asymptotic $\langle x^2\rangle\sim t^2$ regime of the quantum diffusion realized in the commensurate case. Figure 5 shows the velocity v as a function of $T_B/T=r/q$ for the rational numbers with $q<30$. One observes a highly structured behavior which depends sensitively on the denominator q (note that velocities with the same denominator are approximately equal). Up to now we have no analytical estimate for the mean velocity, but nu-

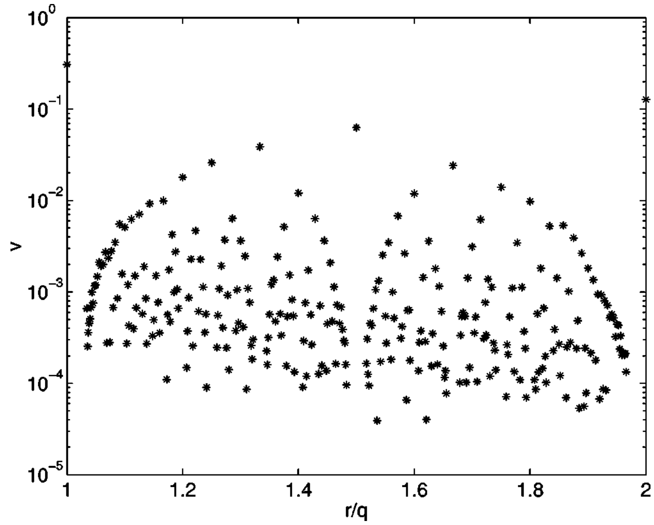

FIG. 5. The mean squared velocity (8) as a function of $T_B/T = r/q$, for rational number in the interval $1 \leqslant r/q \leqslant 2$, $q < 30$.

merical data suggest that v decays (with increasing noncommensurability) faster than $v \sim q^{-3/2}$. This problem requires further study.

IV. CONCLUDING REMARKS

Finally, we would like to discuss the relation of the reported results to those for the kicked quantum rotor [7] and to those of Ref. [17].

For the kicked rotor system, we have classical diffusion of the momentum, which becomes either suppressed or accelerated, depending on the ratio (rational/irrational) of the driving period to some internal quantum period. For the biased kicked Harper system considered here, we find diffusion of the coordinate which is suppressed or accelerated in the quantum case, depending on the ratio of the driving period to the Bloch period (which is a pure quantum quantity). This similarity raises the question whether such an analogy between these two problems can be extended. In particular, it would be interesting to study the dependence of the localization length of the quasienergy wave functions on the classical diffusion coefficient, a problem, which has been intensely discussed for the kicked rotor and variety of the other systems with classical diffusion of the momentum.

The system considered in Ref. [17] is formally closer to the biased kicked Harper (than the kicked rotor) but has a completely different dynamics. Indeed, in the cited paper we have studied the dynamical and spectral properties of a system with the Hamiltonian

$$H = \hat{p}^2/2 + V(x,t) + Fx, \tag{9}$$

where $V(x,t)$ is periodic both in space and time. For system (9) the condition of commensurability between the driving and Bloch periods mainly affects the stability properties. In particular, it was shown that the survival probability (the absolute square of the wave function overlap for $t=0$ and $t>0$) decays algebraically as t^{-q} for $T_B/T = r/q$ rational, in contrast to an exponential decay for the irrational case $T_B/T \neq r/q$. In this present paper, we have considered a ''stabilized version'' of system (9), which is obtained by imposing a periodicity condition on the momentum, $p^2/2 \to -\cos p$. (As mentioned in the introductory part of Sec. II, this situation can be realized for superlattices.) In this case, the condition of commensurability controls the process of chaotic diffusion—for rational T_B/T the diffusion is accelerated but it is suppressed for irrational T_B/T. It is interesting to note that in terms of the survival probability the behavior of system (2) appears to be *opposite* to that of system (9).

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